JO BOALER

FOREWORD BY CAROL DWECK

MATHEMATICAL MINDSETS

Unleashing Students' POTENTIAL Through Creative Math, Inspiring Messages and INNOVATIVE TEACHING

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Case 2. Growing Shapes: The Power of Visualization

The next case I want to share comes from a very different setting—a middle school classroom in a San Francisco Bay Area summer school where students had been referred because they were not performing well in the school year. I was teaching one of the four math classes with my graduate students at Stanford. We had decided to focus the classes on algebra, but not algebra as an end point, with students mindlessly solving for x. Instead, we taught algebra as a problem-solving tool that could be used to solve rich, engaging problems. The students had just finished sixth and seventh grades, and most of them hated math. Approximately half the students had received a D or an F in their previous school year (for more detail, see Boaler, 2015a; Boaler & Sengupta-Irving, 2015).

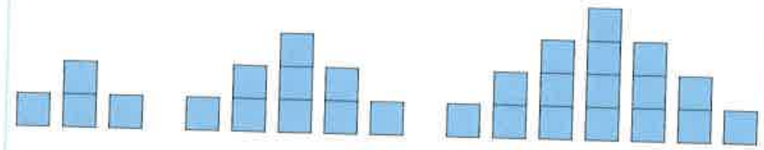
In developing a curriculum for the summer school, we drew upon a range of resources, including Mark Driscoll's books, Ruth Parker's mathematics problems, and two curricula from England—SMILE (which stands for secondary mathematics individualized learning experience) and Points of Departure. The task that created this case of mathematics excitement came from Ruth Parker; it asked the students to extend the growing pattern shown in Exhibit 5.1, made out of multilink cubes, to find how many cubes there would be in the 100th case. (Full-page task worksheets of all exhibits can be found in the Appendix.)

The students had multilink cubes to work with. In our teaching we invited the students to work together in groups to discuss ideas, sometimes groups we teachers chose, other times groups the students chose. On the day in question, I noticed an interesting grouping of three boys—three of the naughtiest boys in my class! They did not know each other before coming to the summer school, but all three spent most of the first week of summer school either off task or working to pull others off task. The boys would shout things out when others were at the board showing math and generally seemed more interested in social conversations than math conversations in the early days. Jorge had received an F in his last math class, Carlos a D, and Luke an A. But the day we gave the students this task, something changed. Incredibly, the three boys worked on this math task for 70 minutes, without ever stopping, becoming distracted, or moving off task. At one point some girls came over and poked them with pencils, which caused the boys to pick up their work and move to another table, they were so intensely engaged in the task and working to solve the problem.

All of our lessons were videotaped, and when we reviewed the film of the boys working that day we watched a rich conversation about number patterns, visual growth, and algebraic generalization. Part of the reason for the boys' intense engagement was an adaptation to the task that we had used—an adaptation that can be used with any math task. In classrooms, typically when function tasks such as the one we gave to the students are assigned, they are usually given with the instruction to find the 100th case (or some other high number) and ultimately the nth case. We did not start with this. Instead, we asked the students to first think alone, before moving to group work, about the ways they *saw* the shape growing. We encouraged them to think visually, not with numbers, and to sketch in their journals, showing us where they saw the extra cubes in each case. The boys saw the growth of the shape in different ways. Luke and Jorge saw the growth ... the bottom of the shape each time; this later became known by the class as

Shapes Task

How do you see the shapes growing?



Source: From Ruth Parker; a task used in MEC courses.

Exhibit 5.1

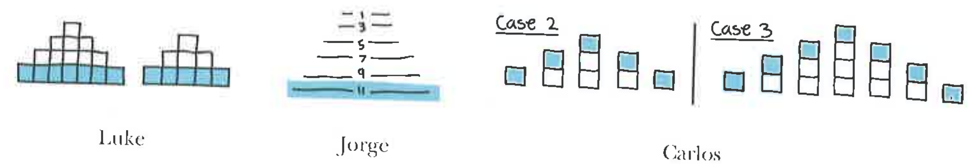


FIGURE 5.2 Students' work
Source: Selling, 2015.

the "bowling alley method," as the cubes arrived like a new line of pins in a bowling alley. Carlos saw the growth as cubes added to the top of the columns—what became known as the "raindrop method"—cubes dropping down from the sky, like raindrops, onto the columns (see Figure 5.2).

After the boys had spent time working out the function growth individually, they shared with each other their ideas for how the shape was growing, talking about where they saw the additional cubes in each case. Impressively, they connected their visual methods with the numbers in the shapes, not only working with their own methods but taking the time to explain the different methods to each other and using each other's methods. The three boys were intrigued by the function growth and worked hard to think about the 100th case, armed with their knowledge of the visual growth of the shape. They proposed ideas to each other, leaning across the table and pointing to their journal sketches. As is typical for mathematical problem solving, they zigzagged around, moving close to a solution, then further away, then back toward it again (Lakatos, 1976). They tried different pathways to the solution, and they broadly explored the mathematical terrain.

I have shown a video of the boys working to many conference audiences of teachers, and all have been highly impressed with the boys' motivation, perseverance, and high-level mathematical conversation. Teachers know that the perseverance shown by the three boys and the respectful ways they discussed each others' ideas, particularly in the context of summer school, is highly unusual, and they are curious as to how we were able to bring it about. They know that many students, particularly those who have been unsuccessful, give up when a task is hard and they don't immediately know the answer. That didn't happen in this looked back at their diagrams and tried out ideas with each other, many of which were incorrect but helpful in ultimately forming a pathway to the solution. After watching the case with teachers at conferences, I ask them what they see in the boys' interactions that could help us understand their high level of perseverance and engagement. Here are some important observations that reveal opportunities to improve the engagement of all students:

1) **The task is challenging but accessible.** All three boys could access the task, but it provided a challenge for them. It was at the perfect level for their thinking. It is very hard to find tasks that are perfect for all students, but when we open tasks and make them broader—when we make them what I refer to as "low floor, high ceiling"—this becomes possible for all students. The floor is low because anyone can see how the shape is growing, but the ceiling is high—the function the boys were exploring is a quadratic function whereby case n can be represented by $(n+1)^2$ blocks. We made the floor of the task lower by inviting the students to think visually about the case—although, as I will discuss later, this was not the only reason for this important adaptation.

2) **The boys saw the task as a puzzle,** they were curious about the solution, and they wanted to solve it. The question was not "real world" or about their lives, but it completely engaged them. This is the power of abstract mathematics when it involves open thinking and connection making.

3) **The visual thinking about the growth of the task gave the boys understanding of the way the pattern grew.** The boys could see that the task grew as a square of (n+1) side length because of their visual exploration of the pattern growth. They were working to find a complex solution, but they were confident in doing so, as they had been given visual understanding to help them.

4) The boys were encouraged by the fact that **they had all developed their own way of seeing the pattern growth** and their different methods were valid and added different insights into the solution. The boys were excited to share their thinking with each other and use their own and each other's ideas in the solving of the problem.

5) **The classroom had been set up to encourage students to propose ideas without being afraid of making mistakes.** This enabled the boys to keep going when they were "stuck," by providing ideas, right or wrong, that enabled the conversation to continue.

6) We had taught the students to **respect each other's thinking.** We did this by valuing the breadth of thinking everyone could offer, not just the procedural thinking that some could offer, valuing the different ways people saw problems and made connections.

7) The students were **using their own ideas,** not just following a method from a book as they learned core algebraic content. The fact that they had proposed different visual ideas for the growth of the function made them more invested and interested in the task.

8) **The boys were working together;** the video shows clearly the way the boys built understanding through the different ideas they shared in conversation, which also enhanced their enjoyment of the mathematics.

9) The boys were working heterogeneously. Viewers of the video note that each boy offers something different and important. The high achiever keeps shouting out number guesses—something that may have been a successful strategy with more procedural questions—but the lower-achieving boys push him to think visually and ultimately more conceptually, and it is the combination of the different boys' thinking that ultimately helps them and leads to success.

Typically, growth pattern tasks are given to students with a numerical question such as "How many cubes are in the 100th case?" and "How many cubes are in the nth case?" We also asked students these questions, but we prefaced them with individual time in which students considered the visual growth of the shape. That changed everything. People think about the growth of the shape in many different ways, as shown in Figures 5.3 through 5.10. When we don't ask students to think visually, we miss an incredible opportunity to increase their understanding. These are some of the ways teachers and students I have worked with see the growth of the shape, accompanied by names they use to capture the growth.

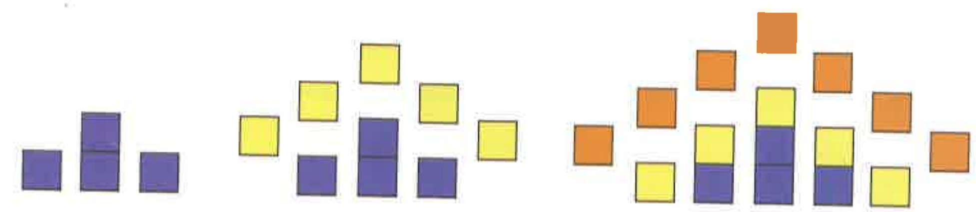


FIGURE 5.3 The Raindrop Method—cubes come from the sky like raindrops

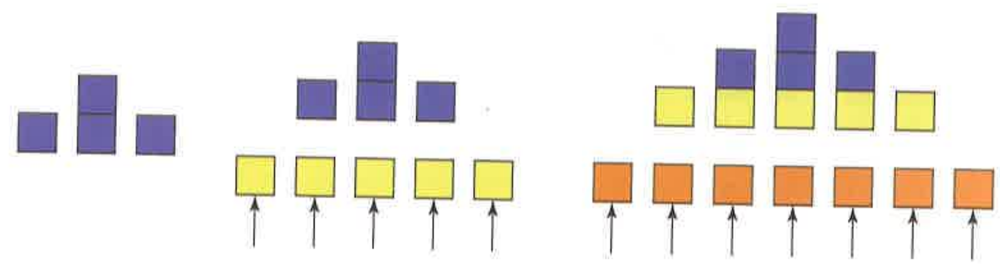


FIGURE 5.4 The Bowling Alley Method—cubes are added like pins in a bowling alley

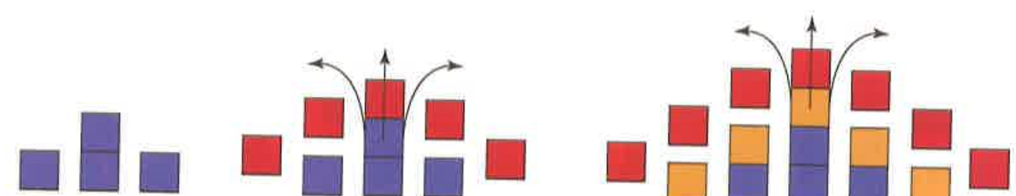


FIGURE 5.5 The Volcano Method—the middle column of cubes grows high and the rest follow like lava erupting from a volcano

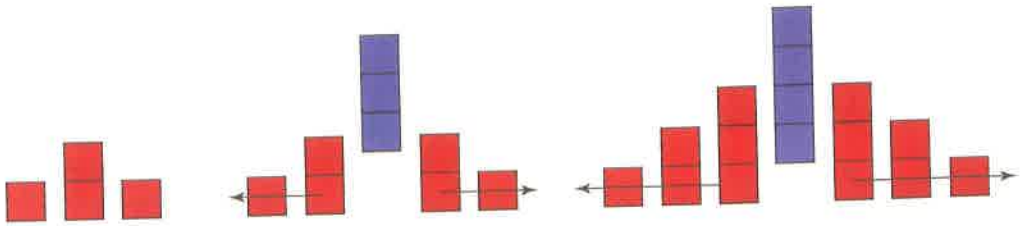


FIGURE 5.6 The Parting of the Red Sea Method—the columns part and the middle column arrives

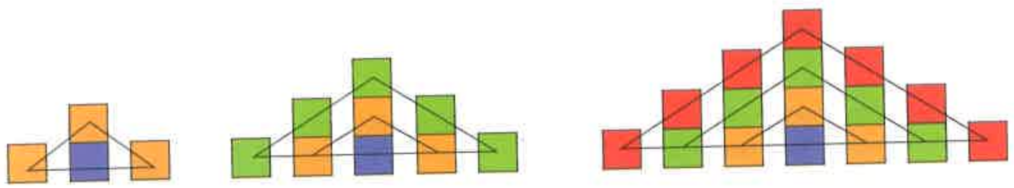


FIGURE 5.7 The Similar Triangles Method—the layers can be seen as triangles

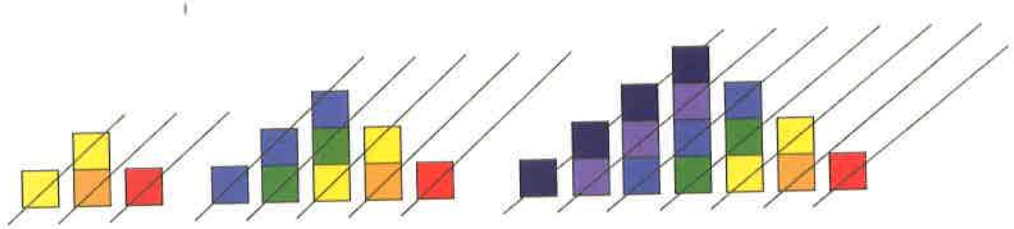


FIGURE 5.8 The Slicing Method—the layers can be viewed diagonally

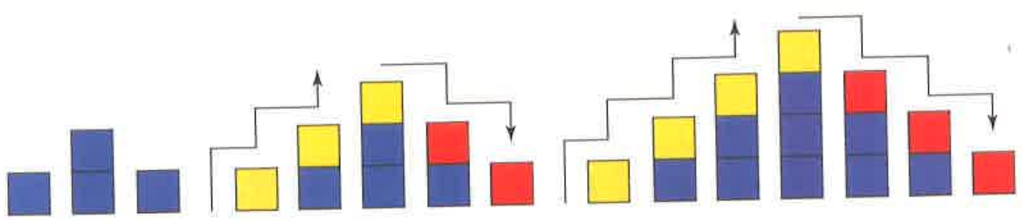


FIGURE 5.9 "Stairway to Heaven, Access Denied"—from *Wayne's World*

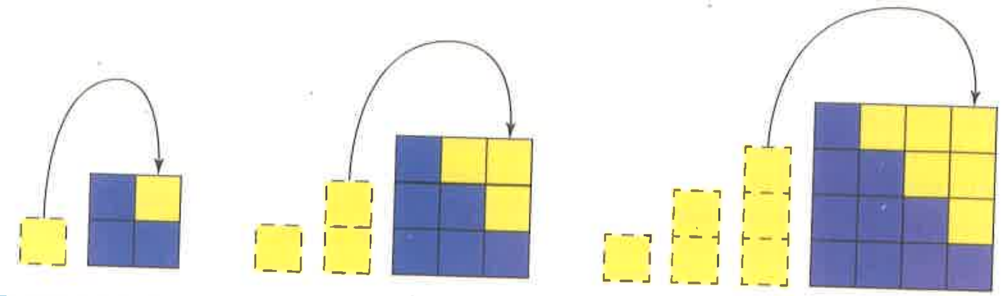


FIGURE 5.10 The Square Method—the shapes can be rearranged as a square each time

I recently gave this pattern-growth task to a group of high school teachers who did not take the time to explore the visual growth of the shape and instead produced a table of values like this:

case	#cubes
1	4
2	9
3	16
n	$(n+1)^2$

When I asked the teachers to tell me why the function was growing as a square, why it was (n+1) *squared*, they had no idea. But this is why we see a squared function: the shape grows as a square, with a side of (n+1), where n is the case number (see Figure 5.11).

When we do not ask students to think visually about the growth of the shape, they do not have access to important understanding about functional growth. They often cannot say what "n" means or represents, and algebra remains a mystery to them—a set of abstract letters they

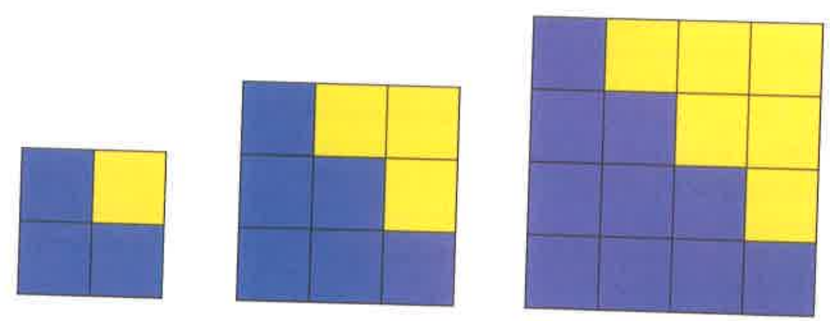


FIGURE 5.11 The Square Method 2

move around on a page. Our summer school students knew what "n" represented, because they had drawn it for themselves. They knew why the function grew as a square and why the nth case was represented by $(n+1)^2$. The algebraic expression they ultimately produced was meaningful to them. Additionally, students did not think they were finding a standard answer for us; they thought they were exploring methods and using their own ideas and thoughts, which included their own ways of seeing mathematical growth. In the final section of this chapter, I will review the ways these features of this task can be used in other tasks to produce increased student engagement and understanding.

Case 3. A Time to Tell?

When I share open, inquiry-based mathematics tasks with teachers, such as the growing shapes or "raindrop" task just discussed, they often ask questions such as, "I get that these tasks are engaging and create good mathematical discussions, but how do students learn new knowledge, such as trig functions? Or how to factorize? They can't discover it." This is a reasonable question, and we do have important research knowledge about this issue. It is true that while ideal mathematics discussions are those in which students use mathematical methods and ideas to solve problems, there are times when teachers needs to introduce students to new methods and ideas. In the vast majority of mathematics classrooms, this happens in a routine of teachers showing methods to students, which students then practice through textbook questions. In better mathematics classrooms, students go beyond practicing isolated methods and use them to solve applied problems, but the order remains—teachers show methods, then students use them.

In an important study, researchers compared three ways of teaching mathematics (Schwartz & Bransford, 1998). The first was the method used across the United States: the teacher showed methods, the students then solved problems with them. In the second, the students were left to discover methods through exploration. The third was a reversal of the typical sequence: the students were first given applied problems to work on, even before they knew how to solve them; then they were shown methods. It was this third group of students who performed at significantly higher levels compared to the other two groups. The researchers found that when students were given problems to solve, and they did not know methods to solve them, but they were given opportunity to explore the problems, they became curious, and their brains were primed to learn new methods, so that when teachers taught the methods, students paid greater attention to them and were more motivated to learn them. The researchers published their results with the title "A Time for Telling," and they argued that the question is not "Should we *tell* or explain methods?" but "When is the best time do this?" Their study showed clearly that the best time was *after* students had explored the problems.

How does this work in a classroom? How do teachers give students problems that they cannot solve without the students experiencing frustration? In describing how this works in practice, I will draw from two different cases of teaching.

The first example comes from the research study I conducted in England that showed that students who learned through a project-based approach achieved at significantly higher

levels in mathematics, both in standardized tests (Boaler, 1998) and later in life (Boaler, 2005), than students who worked traditionally. In one of the tasks I observed in the project school, a group of 13-year-old students were told that a farmer wanted to make the largest enclosure she could out of 36 1-meter pieces of fencing. The students set about investigating ways to find the maximum area. Students tried different shapes, such as squares, rectangles, and triangles, and tried to find a shape with the biggest possible area. Two students realized that the biggest area would come from a 36-sided shape, and they set out to determine the exact area (see Figure 5.12).

They had divided their shape into 36 triangles, and they knew the base of each triangle was 1 meter and the angle at the vertex was 10 degrees (see Figure 5.13).

However, this alone was not enough to find the area of the triangle. At this point the teacher of the class showed the students trigonometry and the ways that a sine function could be used to give them the height of the triangle. The students were thrilled to learn this method, as it helped them solve the problem. I watched as one boy excitedly taught his group members how to use a sine function, telling them he had learned something "really cool" from the teacher. I then remembered the contrasting lesson I had watched in the traditional school a week earlier, in which the teacher had given the students trig functions and then pages of questions to practice them on. In that case, the students had thought the trig functions were extremely boring and unrelated to their lives. In the project school, the students were excited to learn about

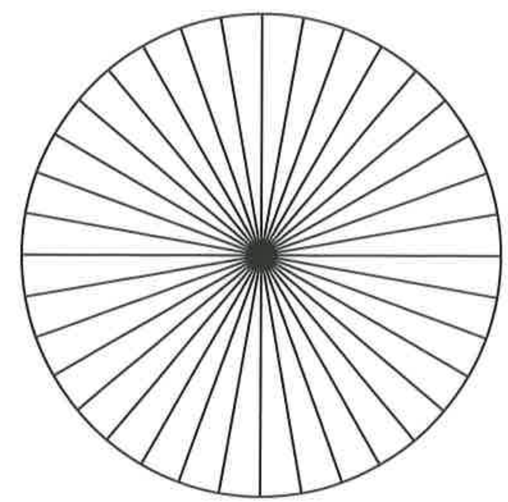


FIGURE 5.12 A 36-sided fence yields the largest enclosure area

FIGURE 5.13 The triangle formed from a 1-meter fence section